

ON On_p

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ABSTRACT. Generalizing John Conway's construction of the Field On_2 , we give the "minimal" definitions of addition and multiplication that turn the ordinals into a Field of characteristic p , for any prime p . We then analyze the structure of the resulting Field, which we will call On_p .

In Chapter 6 of [2], John Conway introduces the Field On_2 : the Class of all ordinals with the appropriate addition and multiplication defined to obtain an algebraically closed Field of characteristic 2. (Following Conway's convention, we will use capitalized terms like "Group", "Ring", and "Field" when the structure referred to is a proper Class.) The operations are referred to as Nim-addition and Nim-multiplication, due to their connections with the game of Nim. Further descriptions of the structure of this Field are given by Lenstra in [5] and [6].

On pg. 17 of [5], Lenstra gives an addition operation which turns the Class of ordinals into an abelian group of exponent 3. Lenstra then asks if there is an analogous definition of multiplication that produces a Field of characteristic 3, and if other characteristics can similarly be handled. In [4], Francois Laubie gives an appropriate definition of addition for any prime characteristic p . (The definition is confined to the finite ordinals, but it works just as well for all ordinals.) In this paper, we will provide definitions of addition and multiplication that turn the ordinals into a Field of any prime characteristic p , which we will call On_p . We will also analyze the structure of these Fields On_p , obtaining results analogous to those in all the aforementioned references.

In what follows, addition and multiplication will always be taken to be operations in On_p for some given prime p . If the standard ordinal operations are needed instead, then the given expression will be enclosed in brackets. For example, $[4 * 4 + 3] = 19$, but in On_2 ,

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$4 * 4 + 3 = 6 + 3 = 5$. We will sometimes use exponentiation in a similar way: $[4^2 + 3] = 19$, but in On_2 , $4^2 + 3 = 4 * 4 + 3 = 5$.

1. ADDITION AND MULTIPLICATION IN On_p

In [2], the definitions of addition and multiplication in On_2 are given "genetically", as follows: for $\alpha, \beta \in On_2$, we have

$$(1) \quad \alpha + \beta = \text{mex}\{\alpha' + \beta, \alpha + \beta'\}$$

and

$$(2) \quad \alpha\beta = \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}.$$

Here, α' ranges over all ordinals less than α , β' ranges over all ordinals less than β , and "mex" represents the "minimal excludent" of the given set. (That is, the smallest ordinal not in that set.)

In [4], a similar genetic definition is given for addition in On_p . Unfortunately, the problem of finding a genetic definition for multiplication in On_p is still open. Rather than working with genetic definitions, we will establish the structure of On_p inductively, defining addition and multiplication on progressively larger fields.

Following the convention in [2], ordinals in On_p will sometimes be treated as single elements of On_p , and will sometimes be treated as the set of all lesser ordinals. Thus, a single ordinal $\alpha \in On_p$ will be called a "group", or "ring", or "field", whenever the set of ordinals $\beta < \alpha$ forms a group, or ring, or field.

For any given ordinal Δ , the " Δ -th" field in On_p will be denoted by ϕ_Δ . In constructing these fields we must ensure that:

- For every ordinal Δ , the ordinal ϕ_Δ is a field of characteristic p .
- If $\alpha < \beta$, then $\phi_\alpha < \phi_\beta$.
- The operations on each field extend those of all previous fields. That is, if $\Delta' < \Delta$, and $\alpha, \beta \in \Delta'$, then $\alpha + \beta$ and $\alpha\beta$ are the same in both $\phi_{\Delta'}$ and ϕ_Δ .

We will construct the fields by induction on Δ , as follows:

- If $\Delta = 0$, then $\phi_\Delta = \phi_0 = p$, and the operations on p are just ordinary addition and multiplication modulo p . Thus, ϕ_0 is isomorphic to \mathbb{F}_p , the finite field of p elements.
- If Δ is a successor ordinal, then we will construct ϕ_Δ from $\phi_{[\Delta-1]}$, using the methods discussed below.
- If Δ is a limit ordinal, then let $\Delta[i]$ be a fundamental sequence of Δ . Then ϕ_Δ will be the limit ordinal whose fundamental sequence is the following: $\phi_\Delta[i] = \phi_{\Delta[i]}$. (In other words, ϕ_Δ is the supremum of all previous fields.)

When Δ is a limit ordinal, the operations on ϕ_Δ are already determined by induction: if $\alpha, \beta \in \phi_\Delta$, then we have $\alpha, \beta \in \phi_{\Delta[i]}$ for some ordinal $\Delta[i]$ in the fundamental sequence of Δ . Then $\alpha + \beta$ and $\alpha\beta$ are defined to be the same in ϕ_Δ as in $\phi_{\Delta[i]}$. Trivially, since every $\phi_{\Delta[i]}$ is a field of characteristic p , so is ϕ_Δ .

So the only remaining work is to define ϕ_Δ when Δ is a successor ordinal. For simplicity in what follows, we will let $\tilde{\phi} = \phi_\Delta$, and we will let ϕ be the previous field: $\phi = \phi_{[\Delta-1]}$.

1.1. When ϕ is not algebraically closed. Assume first that the field ϕ is not algebraically closed. Let n be the smallest positive integer where not all polynomials in $\phi[x]$ of degree n have roots in ϕ . Let $h(x) \in \phi[x]$ be the "lexicographically earliest" polynomial where $g(x) = x^n - h(x)$ has no root in ϕ . In determining which polynomial is lexicographically earliest, we take the coefficients of the largest power of x first. (For example, $5x^3 + 2x^2 + 9x + 17$ is lexicographically earlier than $5x^3 + 3x^2 + 1$.) Note that $g(x)$ is then irreducible over ϕ .

We then define the next field to be $\tilde{\phi} = [\phi^n]$. The definitions of addition and multiplication on $\tilde{\phi}$ will be chosen so that $\tilde{\phi}$ is the extension of the field ϕ by a root of $g(x)$; the ordinal ϕ itself will serve as a root of $g(x)$.

Let F be the factor ring $\phi[x]/\langle g(x) \rangle$. Because $g(x)$ is irreducible over ϕ , F is a field. Every element of F is of the form $f(x) + \langle g(x) \rangle$, where $f(x)$ is a polynomial in $\phi[x]$ of degree less than n . That is, if $a \in F$, then $a = (\sum_{i=0}^{n-1} x^i \alpha_i) + \langle g(x) \rangle$ for some ordinals $\alpha_i \in \phi$. Also, every element of ϕ has a similar representation: if $\alpha \in \tilde{\phi}$, then $\alpha = [\sum_{i=0}^{n-1} \phi^i \alpha_i]$ for some ordinals $\alpha_i \in \phi$. We thus have a one-to-one, onto map between $\tilde{\phi}$ and F : we can define $\theta : \tilde{\phi} \rightarrow F$ via $\theta([\sum_{i=0}^{n-1} \phi^i \alpha_i]) = \sum_{i=0}^{n-1} x^i \alpha_i + \langle g(x) \rangle$.

We will use this map to directly define addition and multiplication in $\tilde{\phi}$: if $\alpha, \beta \in \tilde{\phi}$, then we let $\alpha + \beta = \theta^{-1}(\theta(\alpha) + \theta(\beta))$, and $\alpha\beta = \theta^{-1}(\theta(\alpha)\theta(\beta))$. We then have $\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta)$ and $\theta(\alpha\beta) = \theta(\alpha)\theta(\beta)$, so θ is an isomorphism. Since F is a field, so is $\tilde{\phi}$.

Finally, note that the given operations on $\tilde{\phi}$ preserve those on ϕ . Given $\alpha, \beta \in \phi$, let $\gamma_1 = \alpha + \beta$, $\gamma_2 = \alpha\beta$ in ϕ . Then in $\tilde{\phi}$, we have $\alpha + \beta = \theta^{-1}(\theta(\alpha) + \theta(\beta)) = \theta^{-1}((\alpha + \langle g(x) \rangle) + (\beta + \langle g(x) \rangle)) = \theta^{-1}(\gamma_1 + \langle g(x) \rangle) = \gamma_1$, and similarly $\alpha\beta = \gamma_2$. So as required, $\tilde{\phi}$ is a field of characteristic p extending the operations of ϕ .

1.2. When ϕ is algebraically closed. Now assume that ϕ is an algebraically closed field. The next field will be $\tilde{\phi} = [\phi^\phi]$, and we will define addition and multiplication so that $\tilde{\phi}$ will be isomorphic to $\phi(x)$, the field of rational functions with coefficients in ϕ .

Note: the smallest field in On_p is $\phi_0 = p$, and all fields ϕ_n (for $n \in \omega$) will be algebraic extensions of ϕ_0 , since no finite fields are algebraically closed. The next field must be $\phi_\omega = \omega$. Thereafter, all fields ϕ_Δ will either be a [power] of the preceding field (when Δ is a successor ordinal) or the supremum of all previous fields (when Δ is a limit ordinal). Thus, all infinite fields in On_p will be [powers] of ω , and hence, limit ordinals. (In what follows, we will need the fact that all algebraically closed fields are limit ordinals.)

Following the notation on pg. 62 in [2], any rational function $f(x) \in \phi(x)$ has a partial fraction expansion

$$(3) \quad f(x) = \sum_i \frac{\beta_i}{(x - \alpha_i)^{n_i+1}} + \sum_j \delta_j x^{m_j}$$

where $\alpha_i, \beta_i, \delta_j \in \phi$ and $n_i, m_j \in \omega$ for every i and j . Also, every element in $\tilde{\phi} = [\phi^\phi]$ has the form $[\sum_i \beta_i \phi^{\omega+\omega\alpha_i+n_i} + \sum_j \delta_j \phi^{m_j}]$, where $\alpha_i, \beta_i, \delta_j \in \phi$ and $n_i, m_j \in \omega$ for each i and j . We thus have the following one-to-one, onto map: $\theta : \tilde{\phi} \rightarrow \phi(x)$, where if $\alpha \in \tilde{\phi}$ and $\alpha = [\sum_i \beta_i \phi^{\omega+\omega\alpha_i+n_i} + \sum_j \delta_j \phi^{m_j}]$, then $\theta(\alpha) = \sum_i \frac{\beta_i}{(x-\alpha_i)^{n_i+1}} + \sum_j \delta_j x^{m_j}$.

We define the operations on $\tilde{\phi}$ the same way as in the case where ϕ is not algebraically closed: if $\alpha, \beta \in \tilde{\phi}$, then we let $\alpha + \beta = \theta^{-1}(\theta(\alpha) + \theta(\beta))$, and $\alpha\beta = \theta^{-1}(\theta(\alpha)\theta(\beta))$. We then have $\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta)$ and $\theta(\alpha\beta) = \theta(\alpha)\theta(\beta)$, so θ is an isomorphism. Since $\phi(x)$ is a field, so is $\tilde{\phi}$. And since $\theta(\alpha) = \alpha$ for every $\alpha \in \phi$, the operations in $\tilde{\phi}$ extend those in ϕ . So as required, $\tilde{\phi}$ is a field of characteristic p extending the operations of ϕ .

We now have defined addition and multiplication operations that turn the ordinals into On_p , a Field of characteristic p . But we still must show that these Fields, as defined, are the correct analogues of Conway's field On_2 . We must show that these definitions of addition and multiplication are the "minimal" definitions which will turn the ordinals into a Field of characteristic p .

2. THE MINIMALITY OF On_p

As stated, the definitions of addition and multiplication in On_2 are given in [2] as follows: $\alpha + \beta = \text{mex}\{\alpha' + \beta, \alpha + \beta'\}$, and $\alpha\beta = \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$. These operations can be thought of as the

“minimal” operations that turn the ordinals into a Field. That is, let’s say we tried to write out addition table for the ordinals, not filling in any entry until all lexicographically earlier entries are filled. (So we don’t determine $\alpha + \beta$ until everything of the form $\alpha' + \beta$ or $\alpha + \beta'$ is determined.) For each entry in the table, we always choose the smallest ordinal which allows the resulting structure to be a Field. Then we do the same with multiplication, determining $\alpha\beta$ only after all products of the form $\alpha'\beta$, $\alpha\beta'$, and $\alpha'\beta'$ have been determined. The result would be Conway’s definition of On_2 , for the following reason:

Lemma 2.1. *If F is a Field consisting of the Class of all ordinals, then for all ordinals α, β , we have $\alpha + \beta \geq \text{mex}\{\alpha' + \beta, \alpha + \beta'\}$, and $\alpha\beta \geq \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$.*

So, in On_2 , each sum or product of ordinals yields the smallest ordinal possible, while still ensuring that On_2 is indeed a Field.

Proof. Assume that, for some ordinals α and β , we have $\alpha + \beta < \text{mex}\{\alpha' + \beta, \alpha + \beta'\}$. Then either $\alpha + \beta = \alpha' + \beta$ for some $\alpha' < \alpha$, or $\alpha + \beta = \alpha + \beta'$ for some $\beta' < \beta$. But then either $\alpha = \alpha'$ or $\beta = \beta'$, a contradiction.

Now assume that, for some ordinals α and β , we have $\alpha\beta < \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$. Then for some $\alpha' < \alpha$ and $\beta' < \beta$, we have $\alpha\beta = \alpha'\beta + \alpha\beta' - \alpha'\beta'$. But then $(\alpha - \alpha')(\beta - \beta') = 0$, a contradiction, since fields have no zero divisors. \square

Now, in [4], Laubie gives a genetic definition for p -adic addition in the set of finite ordinals (which can be extended to all ordinals). Under his definition, addition can be done by expressing ordinals in base p , then adding in base p without carrying. (For example, in On_3 , we have $22 + 19 = (9 + 9 + 3 + 1) + (9 + 9 + 1) = 9 + 3 + 1 + 1 = 14$. We will now show that the definitions given in Section 1 produce the same property of addition; thus, our non-genetic definition of addition is the same as Laubie’s genetic definition.

Theorem 2.2. *Given ordinals $\alpha, \beta \in On_p$, assume $\alpha = [\sum p^\delta a_\delta]$ and $\beta = [\sum p^\delta b_\delta]$ (where each δ is an ordinal, and every $a_\delta, b_\delta \in p$). Then $\alpha + \beta = [\sum p^\delta c_\delta]$, where each $c_\delta \equiv a_\delta + b_\delta \pmod{p}$.*

Proof. There exists some field ϕ_Δ containing both α and β . We will proceed by induction on Δ ; the statement is obviously true when $\Delta = 0$, as then $\alpha, \beta \in p$, and addition in p is just ordinary addition modulo p . If Δ is a limit ordinal, then for some ordinal $\Delta[i]$ in the

fundamental sequence of Δ , both α and β are contained in $\phi_{\Delta[i]}$. So by induction, the statement is true in that case.

That leaves the case where Δ is a successor ordinal; let $\tilde{\phi} = \phi_{\Delta}$, and let ϕ be the preceding ordinal. Based on the work in Section 1, we have $\alpha = [\sum \phi^{\delta} \alpha_{\delta}]$ and $\beta = [\sum \phi^{\delta} \beta_{\delta}]$, where each $\alpha_{\delta}, \beta_{\delta} \in \phi$. (If $\tilde{\phi}$ is a degree n extension of ϕ , then every $\delta < n$; if ϕ is algebraically closed, then every $\delta \in \phi$.) For each δ , let $\alpha_{\delta} + \beta_{\delta} = \gamma_{\delta}$; by induction, each of these summations is just componentwise addition modulo p . Then based on the work in Section 1, we have $\alpha + \beta = [\sum \phi^{\delta} \alpha_{\delta}] + [\sum \phi^{\delta} \beta_{\delta}] = \sum \phi^{\delta} \alpha_{\delta} + \sum \phi^{\delta} \beta_{\delta} = \sum \phi^{\delta} \gamma_{\delta} = [\sum \phi^{\delta} \gamma_{\delta}]$. And since ϕ is a [power] of p , this is just componentwise addition modulo p . \square

So indeed, our definition of addition is as it should be. And this leads to one further consequence: the elements of On_p that are groups are exactly the ordinals of the form $[p^{\alpha}]$, for some ordinal α .

We still must show why our definition of multiplication is the correct one.

Definition 2.3. *Given ordinals $\alpha, \beta \in On_p$, we will say that the unordered pair $\{\alpha, \beta\}$ has the "MEX property" if $\alpha\beta = \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$. We will call the set $\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$ the "MEX set" of the unordered pair $\{\alpha, \beta\}$.*

Note: since $\alpha\beta \geq \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$ for all $\alpha, \beta \in On_p$, we can say that $\{\alpha, \beta\}$ has the "MEX property" if $\alpha\beta \leq \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$. In other words, $\{\alpha, \beta\}$ has the "MEX property" if, for all $\gamma < \alpha\beta$, we have $\gamma = \alpha'\beta + \alpha\beta' - \alpha'\beta'$ for some choice of $\alpha' < \alpha, \beta' < \beta$.

With this definition, we can say that $\{\alpha, \beta\}$ has the MEX property for all $\alpha, \beta \in On_2$. But the same does not apply for On_p when $p \geq 3$. The simplest exception: in On_3 , we have $3(3) = 2$ (as we will see later), so $4(4) = (3+1)(3+1) = 2+3+3+1 = 6$. But the minimal excludent of the set $\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$, where α' and β' range over all ordinals less than 4, can be shown to be 2. So we certainly cannot use $\alpha\beta = \text{mex}\{\alpha'\beta + \alpha\beta' - \alpha'\beta'\}$ as a genetic definition of multiplication in On_p .

However, I would conjecture the following:

Conjecture 2.4. *If $\alpha, \beta \in On_p$ are groups, then the pair $\{\alpha, \beta\}$ has the MEX property.*

This conjecture, if true, would establish that we have the minimal possible definition of multiplication: the products of groups are minimal, and those products determine the products of any two ordinals, via the distributive law.

We will not prove this conjecture in general, but we will show that it is true in certain cases. And we will show that these cases determine the product of any two groups in On_p ; hence, they determine the entire multiplication table of On_p . That should be sufficient evidence that our definition of multiplication is the "correct" one.

Lemma 2.5. *If $\phi \in On_p$ is a field, $\beta \in \phi$, $\tilde{\phi}$ is smallest field larger than ϕ , and $[\phi^\alpha] \in \tilde{\phi}$, then the pair $\{[\phi^\alpha], \beta\}$ has the MEX property.*

Proof. Based on the work in Section 1, we have $[\phi^\alpha](\beta) = [\phi^\alpha\beta]$, regardless of whether ϕ is algebraically closed or not. If $\gamma < [\phi^\alpha\beta]$, then $\gamma = [\phi^\alpha\beta_1 + \beta_2] = [\phi^\alpha](\beta_1) + \beta_2$ for some $\beta_1 < \beta$ and some $\beta_2 \in [\phi^\alpha]$. We must show that $\gamma = \alpha_1\beta + [\phi^\alpha]\beta' - \alpha_1\beta'$ for some ordinals $\alpha_1 < \phi^\alpha$, $\beta' < \beta$.

We have $\alpha_1\beta + [\phi^\alpha]\beta' - \alpha_1\beta' = [\phi^\alpha]\beta' + \alpha_1(\beta - \beta')$, and $\alpha_1(\beta - \beta') \in [\phi^\alpha]$. We may choose $\beta' = \beta_1$. And since $\beta, \beta' \in \phi$, and since ϕ is a field, there exists some $\phi' \in \phi$ where $\phi'(\beta - \beta') = 1$. We may then choose $\alpha_1 = \phi'(\beta_2) \in [\phi^\alpha]$. So we have rewritten γ as required; $\{[\phi^\alpha], \beta\}$ has the MEX property. \square

Lemma 2.6. *Let $\phi \in On_p$ be a field that is not algebraically closed, and assume that $\tilde{\phi} \in On_p$, the next larger field, is an extension of ϕ of degree n . Let i, j be nonnegative integers such that $i + j \leq n$. Then the pair $\{[\phi^i], [\phi^j]\}$ has the MEX property.*

Proof. Let $h(x) \in \phi[x]$ be the lexicographically earliest polynomial such that $g(x) = x^n - h(x)$ has no root in ϕ .

If $i + j < n$, then $[\phi^i][\phi^j] = [\phi^{i+j}]$. If $\gamma < [\phi^{i+j}]$, then for some polynomial $m(x) \in \phi[x]$ of degree less than $i + j$, we have $\gamma = [m(\phi)]$. Let $f(x) = x^{i+j} - m(x)$; then $f(x)$ is a monic polynomial of degree less than n . So f can be factored into linear factors over ϕ . Say $f(x) = f_1(x)f_2(x)$, where $f_1(x), f_2(x) \in \phi[x]$, f_1 is monic of degree i , and f_2 is monic of degree j .

Let $m_1(x) = x^i - f_1(x)$ (which has degree less than i), and let $m_2(x) = x^j - f_2(x)$ (which has degree less than j). Then $m(x) = x^{i+j} - f(x) = x^{i+j} - (x^i - m_1(x))(x^j - m_2(x)) = x^i m_2(x) + x^j m_1(x) - m_1(x)m_2(x)$. So we have $\gamma = [\phi^i][m_2(\phi)] + [m_1(\phi)][\phi^j] - [m_1(\phi)][m_2(\phi)]$, and we've rewritten γ in the desired form. So $\{[\phi^i], [\phi^j]\}$ has the MEX property.

Now assume $i + j = n$. Then $[\phi^i][\phi^j] = [h(\phi)]$. If $\gamma < [h(\phi)]$, then $\gamma = m(\phi)$ for some polynomial $m(x) \in \phi[x]$ that is lexicographically earlier than $h(x)$. Let $f(x) = x^{i+j} - m(x)$; then $f(x)$ can be factored into linear factors over ϕ . We then proceed as before; let $f(x) = f_1(x)f_2(x)$, where f_1 is monic of degree i , and f_2 is monic of degree

j . Then if $m_1(x) = x^i - f_1(x)$ (which has degree less than i) and $m_2(x) = x^j - f_2(x)$ (which has degree less than j), then we have $\gamma = [\phi^i][m_2(\phi)] + [m_1(\phi)][\phi^j] - [m_1(\phi)][m_2(\phi)]$. We've rewritten γ in the desired form, so $\{[\phi^i], [\phi^j]\}$ has the MEX property. \square

This essentially establishes that we have the correct definition of multiplication in $\tilde{\phi}$, when the preceding field ϕ is not algebraically closed. If $\tilde{\phi} = [\phi^n]$, then all pairs $\{[\phi^i], [\phi^j]\}$ (where $i + j \leq n$) satisfy the MEX property. And by induction, the products of such pairs determine all products $[\phi^i][\phi^j]$ when $i + j > n$; if $[\phi^i][\phi^j] = \sum_{k=0}^{n-1} \phi^k \gamma_k$ is known, and if $j < n$, then we are forced to have $[\phi^i][\phi^{j+1}] = [\phi^i][\phi^j]\phi = (\sum_{k=0}^{n-1} \phi^k \gamma_k)\phi = \sum_{k=0}^{n-1} \phi^{k+1} \gamma_k$. Finally, if $\alpha, \beta \in \tilde{\phi}$ are groups, then $\alpha = [\phi^i a]$ and $\beta = [\phi^j b]$, where $0 \leq i, j < n$, and $a, b \in \phi$ are groups. Then $\alpha\beta$ is determined by all the aforementioned rules: $\alpha\beta = [\phi^i a][\phi^j b] = [\phi^i][\phi^j](ab) = (\sum_{k=0}^{n-1} \phi^k \gamma_k)(ab) = [\sum_{k=0}^{n-1} \phi^k c_k]$, where $c_k = \gamma_k ab$. So the products of all groups in $\tilde{\phi}$ are determined by the above rules, and thus, so is the entire multiplication table of $\tilde{\phi}$ (by the distributive law).

We still must consider the case where ϕ is algebraically closed.

Lemma 2.7. *If $\phi \in On_p$ is an algebraically closed field, then for all $i, j \in \omega$, the pair $\{[\phi^i], [\phi^j]\}$ has the MEX property.*

The proof is essentially the same as for the $i + j < n$ case in the previous lemma. So, the products $[\phi^i][\phi^j] = [\phi^{i+j}]$ all satisfy the MEX property, and they establish the fact that $[\phi^i] = \phi^i$ for all $i \in \omega$.

Lemma 2.8. *If $\phi \in On_p$ is an algebraically closed field, then for all $\alpha \in \phi$, the pair $\{[\phi^{\omega+\omega\alpha}], \phi\}$ has the MEX property.*

Proof. We have $[\phi^{\omega+\omega\alpha}]\phi = (\frac{1}{\phi-\alpha})\phi = \frac{\alpha}{\phi-\alpha} + 1 = [\phi^{\omega+\omega\alpha}\alpha + 1]$. If $\gamma < [\phi^{\omega+\omega\alpha}\alpha + 1]$, then either $\gamma = [\phi^{\omega+\omega\alpha}\alpha] = \frac{\alpha}{\phi-\alpha}$, or $\gamma = \frac{\alpha'}{\phi-\alpha} + f(\phi)$, where $\alpha' < \alpha$, and $f(x) \in \phi(x)$ is a rational function whose poles are all less than α .

The typical element of the MEX set of $\{[\phi^{\omega+\omega\alpha}], \phi\}$ has the form $g(\phi)\phi + \frac{1}{\phi-\alpha}\phi' - g(\phi)\phi'$, where $\phi' < \phi$, and $g(x) \in \phi(x)$ is a rational function whose poles are all less than α . We may set $\phi' = \alpha'$, obtaining $g(\phi)\phi + \frac{1}{\phi-\alpha}\alpha' - g(\phi)\alpha' = \frac{\alpha'}{\phi-\alpha} + g(\phi)(\phi - \alpha')$. And this will equal $\frac{\alpha'}{\phi-\alpha} + f(\phi)$ when $g(x) = \frac{f(x)}{x-\alpha'}$ (which does only have poles less than α). That covers all possible values of γ except $\gamma = \frac{\alpha}{\phi-\alpha}$, which we may obtain by setting $\phi' = \alpha$ and $g(x) = 0$. So every value of γ is in the MEX set; $\{[\phi^{\omega+\omega\alpha}], \phi\}$ has the MEX property. \square

So, the products $[\phi^{\omega+\omega\alpha}]\phi = [\phi^{\omega+\omega\alpha}]\alpha + 1$ all satisfy the MEX property, and they establish the fact that, for all $\alpha \in \phi$, $[\phi^{\omega+\omega\alpha}] = 1/(\phi - \alpha)$.

Lemma 2.9. *If $\phi \in On_p$ is an algebraically closed field, then for all $\alpha \in \phi$ and all positive integers n , the pair $\{[\phi^{\omega+\omega\alpha+n}], \phi\}$ has the MEX property.*

Proof. We have $[\phi^{\omega+\omega\alpha+n}]\phi = \frac{1}{(\phi-\alpha)^{n+1}}\phi = \frac{1}{(\phi-\alpha)^n} \frac{\phi}{\phi-\alpha} = \frac{1}{(\phi-\alpha)^n} (\frac{\alpha}{\phi-\alpha} + 1) = \frac{\alpha}{(\phi-\alpha)^{n+1}} + \frac{1}{(\phi-\alpha)^n} = [\phi^{\omega+\omega\alpha+n}\alpha + \phi^{\omega+\omega\alpha+n-1}]$. If γ is a smaller ordinal, then γ must take one of the following two forms:

1. $\gamma = \frac{\alpha}{(\phi-\alpha)^{n+1}} + f(\phi)$, where all poles of $f(x) \in \phi(x)$ are at most α , and α is a pole of degree at most $n-1$.
2. $\gamma = \frac{\alpha'}{(\phi-\alpha)^{n+1}} + f(\phi)$, where $\alpha' < \alpha$, all poles of $f(x) \in \phi(x)$ are at most α , and α is a pole of degree at most n .

Meanwhile, the typical element of the MEX set of $\{[\phi^{\omega+\omega\alpha+n}], \phi\}$ has the form $g(\phi)\phi + \frac{1}{(\phi-\alpha)^{n+1}}\phi' - g(\phi)\phi'$, where $\phi' < \phi$, all poles of $g(x) \in \phi(x)$ are at most α and α is a pole of degree at most n .

This typical element of the MEX set will equal $\frac{\alpha}{(\phi-\alpha)^{n+1}} + f(\phi)$ when $\phi' = \alpha$ and $g(x) = \frac{f(x)}{x-\alpha}$. And this typical element of the MEX set will equal $\frac{\alpha'}{(\phi-\alpha)^{n+1}} + f(\phi)$ when $\phi' = \alpha'$ and $g(x) = \frac{f(x)}{x-\alpha'}$. So all values of γ are in the MEX set; $\{[\phi^{\omega+\omega\alpha+n}], \phi\}$ has the MEX property. \square

So, the products $[\phi^{\omega+\omega\alpha+n}]\phi = [\phi^{\omega+\omega\alpha+n}\alpha + \phi^{\omega+\omega\alpha+n-1}]$ all satisfy the MEX property. And they establish the fact that, for all $\alpha \in \phi$, $[\phi^{\omega+\omega\alpha+n}] = 1/(\phi - \alpha)^{n+1}$.

These facts are enough to determine the products of all groups in $[\phi^\phi]$ (and hence, to determine the product of all elements of $[\phi^\phi]$). Any group in $[\phi^\phi]$ has the form $[\phi^\alpha\beta]$, where $\alpha, \beta \in \phi$, and β is a group. We have $[\phi^\alpha\beta] = [\phi^\alpha]\beta$, and based on the above lemmas, $[\phi^\alpha] = f(\phi)$ for some $f(x) \in \phi(x)$. Thus, $[\phi^\alpha]\beta = f(\phi)\beta$, and the product of two such rational functions of ϕ is determined. So multiplication in On_p is completely determined by this collection of products that satisfy the MEX property.

All this raises the question: is there a purely genetic definition of multiplication in On_p ? Even if it turns out that all pairs of groups in On_p satisfy the MEX property, finding a genetic definition of multiplication that works for all elements of On_p would seem to be difficult.

3. THE ALGEBRAIC CLOSURE OF p IN On_p

We will now focus on the structure of On_p below the first transcendental. As we've seen, all ordinals that are fields, but not algebraically closed, will define algebraic extensions of themselves. All ordinals that are algebraically closed fields define transcendental extensions of themselves; if ϕ is an algebraically closed field, then since ϕ is not an element of itself, ϕ must be transcendental over itself! So we will refer to such elements ϕ as "transcendentals" in On_p .

In [2], Conway describes the general structure on On_2 below the first transcendental, $[\omega^{\omega^\omega}]$. For example, the first fields in On_2 are of the form $[2^{2^n}]$, and each is a quadratic extension of the previous one: $2^2 = 3$, $4^2 = 6$, $16^2 = 24$, $256^2 = 384$, and so on. (Each field, when squared, produces the [sesquimultiple] of that field.) The next fields are of the form $[\omega^{3^n}]$, and each is a cubic extension of the previous one: $\omega^3 = 2$, $[\omega^3]^3 = \omega$, $[\omega^9]^3 = [\omega^3]$, etc. Then we have the quintic extensions, and the septic extensions, and so on. In this section, we will see that the fields On_p , for all primes p , each have a similar structure below the first transcendental.

Borrowing the notation in [6], if $r = [u^n]$ is a prime [power] (u prime, $n \in \mathbb{N}$), and if k is the number of primes less than u (so $k = 0$ if $u = 2$, $k = 1$ if $u = 3$, etc.), then we write $\chi_r = [p^{(\omega^k u^{n-1})}]$. Note that we then have $\chi_r = [\omega^{(\omega^{k-1} u^{n-1})}]$ if $k \geq 1$, so there will be no ambiguity in writing χ_r without reference to p as long as $u \geq 3$. (If $u = 2$, and if there is a chance of ambiguity, we will write $\chi_{r,p}$ for $[p^{2^{n-1}}]$.)

Note that we have $\chi_{[u_1^{n_1}]} < \chi_{[u_2^{n_2}]}$ if and only if either $u_1 < u_2$, or $u_1 = u_2$ and $n_1 < n_2$. So, we have $\chi_2 \subseteq \chi_4 \subseteq \chi_8 \subseteq \cdots \subseteq \chi_3 \subseteq \chi_9 \subseteq \cdots \subseteq \chi_5 \subseteq \chi_{25} \subseteq \cdots$. Also, the supremum of all the elements χ_r is $[p^{\omega^\omega}] = [\omega^{\omega^\omega}]$.

Our main goal of this section is to prove the following theorem.

Theorem 3.1. *The following are true for all primes p :*

1. *The first transcendental in On_p is $[\omega^{\omega^\omega}]$.*
2. *The ordinals in $[\omega^{\omega^\omega}] \in On_p$ that are fields are exactly the χ_r for prime [powers] r .*
3. *For each prime u and each natural number $n \geq 2$, $\chi_{[u^n]}$ is a u th degree extension of $\chi_{[u^{n-1}]}$. Also, $\chi_{[u^n]}$ is closed under all extensions of degree u' , for any prime $u' < u$.*
4. *If $p \neq u$, then we have $(\chi_u)^u = \alpha_u$, where α_u is the smallest ordinal in χ_u with no u th root in χ_u .*
5. *We have $(\chi_{[u^{n+1}]})^u = \chi_{[u^n]}$ whenever $p \neq u$ and $n \geq 1$, except in the case where $[u^{n+1}] = 4$ and $p \equiv 3 \pmod{4}$.*

6. We have $(\chi_4)^2 = \chi_2 + 1$ when $p \equiv 3 \pmod{4}$.

7. Finally, we have the case where $p = u$: for $n \geq 2$, we have $(\chi_{[p^n]})^p = \chi_{[p^n]} + \prod_{k=1}^{n-1} (\chi_{[p^k]})^{[p-1]}$. If $n = 1$, then we have $(\chi_p)^p = \chi_p + 1$.

For example, consider the structure of On_3 below the first transcendental. The first extensions are all by square roots: we have $3^2 = 2$, $9^2 = 4$ (not 3, since $3 \equiv 3 \pmod{4}$), $81^2 = 9$, $6561^2 = 81$, and so on. Then we have the cubic extensions; we have $\omega^3 = \omega + 1$, $[\omega^3]^3 = [\omega^3 + \omega^2]$, $[\omega^9]^3 = [\omega^9 + \omega^8]$, and so on. Then come the quintic extensions: $[\omega^\omega]^5 = 10$, $[\omega^{\omega \cdot 5}]^5 = [\omega^\omega]$, $[\omega^{\omega \cdot 25}]^5 = [\omega^{\omega \cdot 5}]$, etc. And this pattern continues throughout all extensions. (In Section 4, we will look at methods for determining the values of α_u for $u \neq p$: $\alpha_2 = 2$, $\alpha_5 = 10$, etc.)

We will prove Theorem 3.1 by a sequence of lemmas.

Lemma 3.2. *Every element of On_p below the first transcendental is contained in a finite field within On_p .*

True by induction: given an element α below the first transcendental, let χ be the smallest ordinal where χ is a field and $\chi > \alpha$. If χ is finite, then χ is the finite field we want; so assume χ is infinite. From the work in Section 1, χ is an algebraic extension of a smaller field χ' , so α is a root of some irreducible polynomial $f(x) \in \chi'[x]$. By induction, each coefficient of $f(x)$ is contained in a finite field, so there is a finite field F containing all the coefficients of $f(x)$. If the degree of $f(x)$ is n , then $F(\alpha)$ will be a field in On_p containing α , and it will have order $|F|^n$, which is finite. So $F(\alpha)$ is the field we want.

Lemma 3.3. *Let $\chi \in On_p$ be a field below the first transcendental. Then for any prime u , the following are equivalent:*

1. *All irreducible polynomials of degree u in $\chi[x]$ have roots in χ .*
2. *χ contains finite fields of order $[p^{u^n}]$ for all $n \in \omega$.*

Proof. $1 \Rightarrow 2$: This can be proven by induction on n . Obviously, since χ is a field, χ contains a field of order $[p^{u^0}] = p$. Assume that $F \subseteq \chi$ is a field of order $[p^{u^k}]$. Let $f(x) \in F[x]$ be an irreducible polynomial of degree u ; this field has a root $\alpha \in \chi$, but then $F(\alpha) \subseteq \chi$ is a field of order $[p^{u^{k+1}}]$.

$2 \Rightarrow 1$: Let $f(x) \in \chi[x]$ be an irreducible polynomial of degree u . Let $F \subseteq \chi$ be a finite field containing all the coefficients of $f(x)$. Say $|F| = [p^{mu^n}]$, where m is not a multiple of u ; then all of the roots of $f(x)$ are in the field of order $[p^{mu^{n+1}}]$. But χ contains both a field of order $[p^{mu^n}]$ (namely, F) and a field of order $[p^{u^{n+1}}]$ (by assumption),

so χ must contain a field of order $[p^{mu^{n+1}}]$. So χ contains a root of $f(x)$. \square

Lemma 3.4. *Let $\chi \in On_p$ be a field below the first transcendental. Let u_1, u_2, \dots, u_m be primes. Then the following are equivalent:*

1. *All irreducible polynomials of degree u_i in $\chi[x]$ have roots in χ , for every i .*
2. *If all the prime factors of $n \in \mathbb{N}$ are among the u_i , then all irreducible polynomials of degree n in $\chi[x]$ have roots in χ .*

Proof. $2 \Rightarrow 1$: trivial, as we can just take $n = u_i$ for each u_i in turn.

$1 \Rightarrow 2$: let $f(x) \in \chi[x]$ be an irreducible polynomial of degree n . Let $F \subseteq \chi$ be a finite field containing all the coefficients of $f(x)$. Then all the roots of $f(x)$ are in the finite field of order $|F|^n$. But this field can be built up from F using extensions of prime degree (i.e. of degree u_i for some i), and by assumption, the fields resulting from each extension will be contained in χ . So the roots of $f(x)$ will be in χ . \square

Essentially, this proves parts 1 through 3 of Theorem 3.1. Below the first transcendental, the first fields will define quadratic extensions of themselves. If ϕ is a field, and the next field $\tilde{\phi}$ is a quadratic extension, then we have seen that $\tilde{\phi} = [\phi^2]$. So the first fields are p , $[p^2]$, $[p^4]$, and so on. The supremum of these fields ($[p^\omega] = \omega$) is quadratically closed, and the subsequent fields $[\omega^3]$, $[\omega^9]$, etc. will each be cubic extensions of the previous field. Once we have a cubically closed field, then come the quintic extensions, then the septic extensions, and so on. And the supremum of all these fields is the first transcendental, $[\omega^{\omega^\omega}]$.

To prove the next parts of Theorem 3.1, we will introduce some notation from [6]. For $\alpha \in [\omega^{\omega^\omega}]$, we will let $d(\alpha)$ be the degree of the minimal polynomial of α over the field p . (So, the smallest field containing α has order $[p^{d(\alpha)}]$.) Also, if $\alpha \neq 0$, we will let $ord(\alpha)$ be the multiplicative order of α : i.e. the smallest $n \in \mathbb{N}$ where $\alpha^n = 1$.

Lemma 3.5. *For every nonzero $\alpha \in [\omega^{\omega^\omega}]$, $d(\alpha)$ is the smallest $m \in \mathbb{N}$ where $ord(\alpha)$ divides $[p^m - 1]$.*

Proof. Let $n = d(\alpha)$. Then α is contained in a field of order p^n , but not contained in a field of order p^m for any $m < n$. So α is a root in On_p of $x^{[p^n]} - x$ (hence of $x^{[p^n-1]} - 1$), but not of $x^{[p^m]} - x$ (nor of $x^{[p^m-1]} - 1$) for any $m < n$. So the multiplicative order of α is a factor of $[p^n - 1]$, but not of $[p^m - 1]$ for any $m < n$. \square

For simplicity in what follows, we will say that $n \in \mathbb{N}$ is a “primitive divisor” of $[p^m - 1]$ (for $m \in \mathbb{N}$) if n divides $[p^m - 1]$, but n does not divide $[p^i - 1]$ for any $i \in \mathbb{N}$, $i < m$. (Any factor of $[p^1 - 1]$ is automatically a primitive divisor of $[p^1 - 1]$.) Thus, we have proven that for every nonzero $\alpha \in [\omega^{\omega^\omega}]$, $\text{ord}(\alpha)$ is a primitive divisor of $[p^{d(\alpha)} - 1]$.

Also, if $u, m, n \in \mathbb{N}$, we will write $[u^m] || n$ if $[u^m]$ divides n , but $[u^{m+1}]$ does not divide n . (If u does not divide n , then we will write $[u^0] || n$.)

Lemma 3.6. *Say $F \subseteq On_p$ is a field of order $[p^n]$, $\alpha \in F$, and u is a prime. Assume $[u^m] || [p^n - 1]$. Then α is an u th power in F (i.e. there exists a $\beta \in F$ where $\beta^u = \alpha$) if and only if one of the following occurs:*

1. $m = 0$ (so $[p^n - 1]$ is not a multiple of u)
2. $[u^m]$ does not divide $\text{ord}(\alpha)$.

Proof. If $m = 0$, then there exists $k \in \mathbb{N}$ such that $[ku] \equiv 1 \pmod{[p^n - 1]}$. Let $\beta = \alpha^k \in F$; then $\beta^u = \alpha^{[ku]} = \alpha$.

If $m > 0$, then assume $\text{ord}(\alpha) = [u^k s]$, where u is not a factor of s . Then $k \leq m$. Let $\beta \in On_p$ be a u th root of α ; then $\text{ord}(\beta) | [u^{k+1} s]$. If $k < m$, then $\text{ord}(\beta) | [p^n - 1]$, so $\beta \in F$. If $k = m$, then since $[u^m]$ divides $\text{ord}(\alpha)$, $[u^{m+1}]$ must divide $\text{ord}(\beta)$. So $\text{ord}(\beta)$ does not divide $[p^n - 1]$, so $\beta \notin F$. \square

Among other things, this shows that below the first transcendental, no p th degree extensions in On_p will be by p th roots. The reason: if $\alpha \in [\omega^{\omega^\omega}]$ is contained in a field of order $[p^n]$, then α is already a p th power in that finite field, since $[p^n - 1]$ is not a multiple of p . But as we will see, all extensions of degree u (where u is a prime other than p) will be by u th roots.

We will need one more lemma before proving parts 4 through 6 of Theorem 3.1. This number-theoretic lemma is Lemma 2.3 in [3].

Lemma 3.7. *If u is a prime, and s, n are natural numbers ($s \geq 2$) such that $u | [s - 1]$, then with one exception, $u^a || [s^n - 1]$ iff $u^a || [n(s - 1)]$. The exception: if $u = 2$, n is even, and $s \equiv 3 \pmod{4}$, then $u^a || [s^n - 1]$ iff $u^a || [n(s + 1)]$.*

Among other things, this means that if u divides $[s - 1]$ but does not divide n , then the u -part of $[s - 1]$ equals the u -part of $[s^n - 1]$. (This fact will be needed in the proof of part 4 of Theorem 3.1.)

We can now prove part 4 of Theorem 3.1.

Lemma 3.8. *For each prime $u \neq p$, there exists an element in χ_u that has no u th root in χ_u . Thus, if α_u is the smallest such element, then $\chi_u^u = \alpha_u$.*

Proof. Assume that u is a primitive divisor of $[p^m - 1]$; then all prime factors of m are less than u . Say $[u^n] || [p^m - 1]$. Let $F \subseteq \chi_u$ be a finite field; then $|F| = [p^s]$, where all prime factors of s are less than u . Then $u | [p^s - 1]$ if and only if s is a multiple of m ; if that is true, then since u does not divide s , the u -part of $[p^s - 1]$ is the same as the u -part of $[p^m - 1]$ (namely, $[u^n]$). So, there are elements of χ_u of multiplicative order $[u^n]$, but no elements of multiplicative order $[u^{n+1}]$.

Let $E \subseteq \chi_u$ be a field of order $[p^m]$ (one such exists, since all prime factors of m are less than u). Let $\alpha \in E$ be an element where $\text{ord}(\alpha) = [u^n]$. Assume that there does exist $\beta \in \chi_u$ where $\beta^u = \alpha$. Then let $E' = E(\beta)$; $|E'| = [p^s]$, where s is a multiple of m . Then $[u^n] || [p^s - 1]$; we cannot have $[u^{n+1}] || [p^s - 1]$, since χ_u has no elements of order $[u^{n+1}]$. So by the previous lemma, α does not have a u th root in E' (since $[u^n]$ does divide the order of $\text{ord}(\alpha)$). So we have a contradiction; there is no $\beta \in \chi_u$ that is a u th root of α . \square

Now to prove parts 5 and 6 of Theorem 3.1.

Lemma 3.9. *We have $(\chi_{[u^{n+1}]})^u = \chi_{[u^n]}$ whenever $p \neq u$ and $n \geq 1$, with one exception: we have $(\chi_4)^2 = \chi_2 + 1$ when $p \equiv 3 \pmod{4}$.*

Proof. Assume that we are not in the exceptional case: either $u > 2$, or $u = 2$ and $p \equiv 1 \pmod{4}$. By induction, we will assume that $(\chi_{[u^{i+1}]})^u = \chi_{[u^i]}$ whenever $1 \leq i < n$, and we will prove that $(\chi_{[u^{n+1}]})^u = \chi_{[u^n]}$. Note: since $(\chi_{[u^n]})^{[u^n]} = \alpha_u$, if $\text{ord}(\alpha_u) = a$, then $\text{ord}(\chi_{[u^n]}) = [au^n]$. And if $d(\alpha_u) = k$, then $d(\chi_{[u^n]}) = [ku^n]$.

Consider any $\beta < \chi_{[u^{n+1}]}$; we claim that β is a u th power in $\chi_{[u^{n+1}]}$. If β is already a u th power in its minimal field $p(\beta)$, then we're done. Otherwise, if $|p(\beta)| = [p^b]$ (then $d(\beta) = b$), and if $[u^c] || [p^b - 1]$ (then $c > 0$), then $[u^c] || \text{ord}(\beta)$. But then the finite field $p(\beta, \chi_{[u^n]})$ has dimension a multiple of $[bu]$ (say, $[bju]$). And since $[u^{c+1}] || [p^{bju} - 1]$, but $[u^{c+1}]$ does not divide $\text{ord}(\beta)$, β must be a u th power in $p(\beta, \chi_{[u^n]})$. So all ordinals less than $[\chi_{[u^n]}]$ are u th powers in $[\chi_{[u^{n+1}]}]$.

But $[\chi_{[u^n]}]$ is not a u th power in $[\chi_{[u^{n+1}]}]$, for the following reasons: if $d(\alpha_u) = k$, and if $[u^m] || [p^k - 1]$, then $[u^m] || \text{ord}(\alpha)$. So, $[u^{m+n}] || \text{ord}(\chi_{[u^n]})$. Consider any finite field $F \subseteq \chi_{[u^{n+1}]}$ containing $\chi_{[u^n]}$; it has dimension u^nt , where t is a multiple of k , and all prime factors of t are less than u . We have $[u^{m+n}] || [p^{u^{nt}} - 1]$; since $[u^{m+n}] || \text{ord}(\chi_{[u^n]})$ also, $\chi_{[u^n]}$ is not a u th power in any finite field in $\chi_{[u^{n+1}]}$ (hence, it is not a u th power in $\chi_{[u^{n+1}]}$). So $\chi_{[u^n]}$ is the smallest element of $\chi_{[u^{n+1}]}$ that has no u th root in $\chi_{[u^{n+1}]}$; we must have $(\chi_{[u^{n+1}]})^u = \chi_{[u^n]}$.

Now assume $u = 2$ and $p \equiv 3 \pmod{4}$. The same reasoning as before shows that if $\beta < \chi_2 = p$, then β is a square in $\chi_4 = [p^2]$.

But χ_2 will also be a square in χ_4 : if $[2^m] \parallel [p-1]$ and $p \equiv 3 \pmod{4}$, then $[2^{m+2}] \parallel [p^2-1]$. But $[2^{m+1}] \parallel \text{ord}(\chi_2)$. So the largest [power] of 2 dividing $[p^2-1]$ does not divide the order of χ_2 , so χ_2 is a square in the field of order $[p^2]$ (namely, χ_4).

So if $p \equiv 3 \pmod{4}$, then χ_2 is a square in χ_4 . If $\chi_2 + 1$ is a square in χ_4 , then for some $a, b \in p$, we have $\chi_2 + 1 = (a\chi_2 + b)^2 = (a^2\alpha_2 + b^2)\chi_2 + (2 \cdot ab)$. So we have $a^2\alpha_2 + b^2 = 2 \cdot ab$ (both sides of that equation are equal to 1), and by manipulating that equation, we obtain $(b-a)^2 = a^2(1-\alpha)$. Since $(b-a)^2$ and a^2 are squares in χ_2 , $1-\alpha$ must also be a square in χ_2 .

But it is not possible for $1-\alpha$ to be a square in χ_2 , for the following reasons: since α is the smallest non-square in χ_2 , $[\alpha-1] = \alpha-1$ is a square in χ_2 . And since $p \equiv 3 \pmod{4}$, -1 is not a square in χ_2 . So $1-\alpha = (-1)(\alpha-1)$, the product of a non-square and a square in χ_2 , cannot be a square in χ_2 . So there is no element $a\chi_2 + b \in \chi_4$ whose square is $\chi_2 + 1$; we must have $\chi_4^2 = \chi_2 + 1$.

However, the inductive step works thereafter: if $[2^{m'}] \parallel \text{ord}(\chi_2 + 1)$, then $[2^{m'}] \parallel [p^2-1]$, so $[2^{m'+1}] \parallel [p^4-1]$. And since $[2^{m'+1}] \parallel \text{ord}(\chi_4 + 1)$, χ_4 will not be a square in χ_8 . So $\chi_8^2 = \chi_4$, and we may proceed by induction as before. So $(\chi_{[u^{n+1}]})^u = \chi_{[u^n]}$ in all cases, except that $(\chi_4)^2 = \chi_2 + 1$ when $p \equiv 3 \pmod{4}$. \square

It remains to prove part 7 of Theorem 3.1. For simplicity, let $\chi = \chi_p$; then $\chi_{[p^n]} = [\chi^{p^{n-1}}]$ for all $n \in \mathbb{N}$. As discussed, each of these fields is a degree p extension of the previous field, but not by a p th root (since all elements of finite fields of characteristic p already have p th roots in that finite field). So each $\chi_{[p^n]}$ will be an extension of $\chi_{[p^{n-1}]}$ by a roots of a polynomial of form $x^p - x - \alpha$, assuming there is such a polynomial with no roots in $\chi_{[p^n]}$. And as we will see, there will always be such a polynomial; the smallest suitable α will be $\prod_{k=1}^{n-1} (\chi_{[p^k]})^{[p-1]} = [\chi^{p^n-1}]$.

Let $f(x) = x^p - x$; for any field $F \subseteq On_p$, f is a group homomorphism from F to itself. Let S be the set of all ordinals in χ that are [multiples] of p ; that is, all ordinals of the form $[p\delta]$ for some ordinal δ .

Lemma 3.10. *The map $f(x)$ sends the ordinals in χ to exactly the ordinals in S .*

Proof. First, we'll show that only ordinals in S get mapped to. Since f is an additive homomorphism, we only need to show that groups in χ get sent to elements of S . Let α be an arbitrary group in χ ; we might as well assume $\alpha > 1$, since $f(1) = 0 \in S$. Then α has the form

$\chi_r^m \delta$, where $r = [u^n]$ is a [power] of a prime $u < p$, m is a positive integer less than u , and $\delta \in \chi_r$ is a group.

Then $f(\alpha) = f(\chi_r^m \delta) = \chi_r^{[mp]} \delta^p - \chi_r^m \delta = \chi_r^{[au+b]} \delta^p - \chi_r^m \delta$, where a, b are nonnegative integers, and $b < p$. Since $[mp]$ is not a multiple of u , $b > 0$. We then have $\chi_r^{[au+b]} \delta^p - \chi_r^m \delta = \chi_r^b (\chi_r^u)^a \delta^p - \chi_r^m \delta = \chi_r^b (\delta') - \chi_r^m \delta$, where $\delta' = (\chi_r^u)^a \delta^p \in \chi_r$. Since $b, m > 0$, both $\chi_r^b (\delta')$ and $\chi_r^m \delta$ are in S ; thus, so is $\chi_r^b (\delta') - \chi_r^m \delta$. So all groups in χ (and hence, all elements of χ) are mapped by f into S .

It remains to show that f maps χ to the entire set S . Choose an arbitrary $\alpha \in S$, and let F be the smallest finite field containing α . Then f is a group homomorphism from F to itself, and its kernel contains p elements (namely, the ordinals less than p). So the image of f on F contains $[|F|/p]$ elements. But the image must contain only elements of $F \cap S$, and there are exactly $[|F|/p]$ elements in $F \cap S$. So all elements of $F \cap S$, including α itself, must be mapped to by some element of F . So all elements of S are mapped to by some element in χ . \square

This is enough to establish the following:

Corollary 3.11. χ is a root of $x^p - x - 1$.

The reason: all polynomials in $\chi[x]$ of form $x^p - \alpha$ already have roots in χ . So does the polynomial $x^p - x$ (namely, all the ordinals less than p). But $x^p - x - 1$ has no root in χ ; given any $\alpha \in \chi$, $f(\alpha) = \alpha^p - \alpha$ is a [multiple] of p , hence cannot be 1. So χ must be a root of $x^p - x - 1$.

The rest of part 7 of Theorem 3.1 will be established by the following theorem:

Theorem 3.12. For each field $\chi_{[p^{n+1}]} = [\chi^{p^n}]$ (for $n \in \omega$), the image of f on $[\chi^{p^n}]$ is the set $S_n = \{\sum_{k=0}^{[p^n-1]} [\chi^k] \beta_k : \beta_k \in \chi, \beta_{[p^n-1]} \in S\}$. The smallest element of $[\chi^{p^n}]$ that is not in S_n is $[\chi^{p^{n-1}}]$; thus $[\chi^{p^n}]$ is a root of $x^p - x - [\chi^{p^{n-1}}]$.

Proof. We will prove this by induction. We already established the $n = 0$ case, so we will assume it is true for $n - 1$, and prove it is true for n .

For simplicity, let $\phi = [\chi^{p^{n-1}}]$, and let $\tilde{\phi}$ be the next field, $[\chi^{p^n}]$. Let $\alpha \in \tilde{\phi}$; we have $\alpha = \sum_{k=0}^{p-1} \phi^k \alpha_k$, where each $\alpha_k \in \phi$. Then $f(\alpha) = \sum_{k=0}^{[p-1]} f(\phi^k \alpha_k) = \sum_{k=0}^{[p-1]} (\phi^{[kp]} \alpha_k^p - \phi^k \alpha_k) = \sum_{k=0}^{[p-1]} (\phi + \chi^{p^{n-1}-1})^k \alpha_k^p - \phi^k \alpha_k = \phi^{[p-1]} (\alpha_{[p-1]}^p - \alpha_{[p-1]}) + \delta$, where $\delta \in \phi^{[p-1]}$. By the inductive hypothesis, $(\alpha_{[p-1]}^p - \alpha_{[p-1]}) = f(\alpha_{[p-1]}) = \sum_{k=0}^{[p^{n-1}-1]} [\chi^k] \beta_k$, where $\beta_k \in$

$\chi, \beta_{[p^{n-1}-1]} \in S$. Thus, $f(\alpha)$ is equal to $[\chi^{p^n-1}]\beta_{[p^{n-1}-1]}$ plus a sum of "lesser terms"; $f(\alpha) \in S_n$.

To show that the image of f on $[\chi^{p^n}]$ includes the entire set S_n : let $\alpha \in S_n$, and let $\alpha = \sum_{k=0}^{[p^n-1]} [\chi^k] \alpha_k$, where each $\alpha_k \in \chi$. (Then $\alpha_{[p^n-1]} \in S$.) Let F be the smallest finite field containing each α_k and each $[\chi^k]$ for all k from 0 to $[p^n-1]$; then $\alpha \in F$. The mapping f is a group homomorphism from F to itself, and its kernel contains p elements (the ordinals less than p). So the image of f over F must contain $[|F|/p]$ elements. But the image is contained in $F \cap S_n$, and there are exactly $[|F|/p]$ elements in $F \cap S_n$. So the image of f over F is exactly the set $F \cap S_n$, which contains α . So all elements $\alpha \in S_n$ are mapped to by some element of $[\chi^{p^n}]$. \square

That completes the proof of part 7 of Theorem 3.1: for each n , $[\chi^{p^n}]$ is a root of $x^p - x - [\chi^{p^{n-1}}]$, since $[\chi^{p^{n-1}}]$ is the smallest element of $[\chi^{p^n}]$ that is not mapped to by $f(x)$.

In summary, we have now determined the full structure of $[\omega^{\omega^\omega}] \in On_p$, with the exception of the unknown elements $\alpha_u \in On_p$ for each prime $u \neq p$. (We will discuss how to find each α_u in the next section.) The structure is similar for all primes p ; $[\omega^{\omega^\omega}]$ is always the first transcendental, and if $\phi \in On_2$ is an infinite field below $[\omega^{\omega^\omega}]$, then ϕ is a field in every On_p . I would conjecture that this pattern continues beyond $[\omega^{\omega^\omega}]$:

Conjecture 3.13. *If $\phi \in On_2$ is an infinite field, then ϕ is a field in On_p for all primes p . If $\phi \in On_2$ is a transcendental, then ϕ is a transcendental in On_p for all primes p .*

But a proof of this conjecture would seem to be well out of reach. Only one transcendental in On_2 is known: namely, $[\omega^{\omega^\omega}]$. The problem of finding the second transcendental in On_2 is wide open; the problem would seem to be equally difficult in On_p for other primes p .

4. EFFECTIVE COMPUTATION BELOW THE FIRST TRANSCENDENTAL IN On_p

We will now discuss methods for finding the elements $\alpha_u \in On_p$ for each prime $u \neq p$. In [6], it is shown that the elements α_u can be effectively determined in On_2 ; we will show that the same can be done in On_p for every prime p . So multiplication can be done effectively in $[\omega^{\omega^\omega}]$; division can be done effectively as well, since all elements of $[\omega^{\omega^\omega}]$ have finite multiplicative order.

As before, for $\alpha \in [\omega^{\omega^\omega}]$, let $d(\alpha)$ be the degree of the irreducible polynomial of α over the field p . Based on our results from Section 3, we have the following proposition (which is identical to Proposition 1.8 in [6]):

Proposition 4.1. *We have that χ_r (for any prime [power] r) is the smallest element of $[\omega^{\omega^\omega}]$ whose irreducible polynomial has degree divisible by r . In other words, if $r = [u^n]$ (for u prime), then χ_r is the set of all ordinals $\alpha \in [\omega^{\omega^\omega}]$ where $d(\alpha)$ is divisible only by primes $\leq u$ and where r does not divide $d(\alpha)$.*

Following [6], we will extend this notation and define χ_h for all positive integers h ; χ_h is the smallest ordinal $\alpha \in [\omega^{\omega^\omega}]$ where $d(\alpha)$ is divisible by h . (We clearly have $\chi_1 = 0$.)

The following lemma is a generalization of Lemma 2.5 in [6].

Lemma 4.2. *Let $\beta, \gamma \in On_p$ be elements of $[\omega^{\omega^\omega}]$, and let $r = [u^n]$ be a [power] of a prime u . If r divides $d(\beta)$ but not $d(\gamma)$, then r divides both $d(\beta + \gamma)$ and $d(\beta - \gamma)$.*

Proof. We have $\beta \in p(\gamma, \beta + \gamma)$, which is an extension of p of degree $\text{lcm}(d(\gamma), d(\beta + \gamma))$. So $d(\beta)$ divides $\text{lcm}(d(\gamma), d(\beta + \gamma))$.

Since r divides $d(\beta)$, r divides $\text{lcm}(d(\gamma), d(\beta + \gamma))$; since r does not divide $d(\gamma)$, r must divide $d(\beta + \gamma)$.

Similar reasoning shows that r divides $d(\beta - \gamma)$. \square

Lemma 4.2 will be used in proving the following two results; they are generalizations to On_p of Theorem 2.1 and Corollary 2.2 in [6].

Theorem 4.3. *Let $h > 1$ be a natural number, let u be the smallest prime dividing h , let r be the largest [power] of u dividing h , and let $g = [h/r]$. Then $\chi_h = \chi_g$ if r divides $d(\chi_g)$; otherwise, $\chi_h = \chi_g + \chi_r = [\chi_g + \chi_r]$.*

Proof. The theorem is proven by induction on the number of primes dividing h . If $h = r$, then $g = 1$; the theorem holds true in that case, since $\chi_h = 0 + \chi_r = \chi_g + \chi_r$, and r does not divide $d(\chi_g) = d(0) = 1$. So assume h has at least two distinct prime divisors. Note that we must have $\chi_h \geq \chi_g$, since g divides h .

By the inductive hypothesis, χ_g is a finite sum of terms $\chi_{r'}$, where each r' is a [power] of a larger prime than u . Thus, each $\chi_{r'}$ is larger than χ_r . We can thus conclude that, for all $\alpha \leq \chi_r$, we have $\chi_g + \alpha = [\chi_g + \alpha]$. Specifically, we have $\chi_g + \chi_r = [\chi_g + \chi_r]$.

Assume that r does divide $d(\chi_g)$. By definition, g also divides $d(\chi_g)$; thus, h divides $d(\chi_g)$. But χ_h is the smallest element of $[\omega^{\omega^\omega}]$ whose minimal polynomial has degree a multiple of h ; thus, $\chi_g \geq \chi_h$. We already showed $\chi_h \geq \chi_g$, hence they are equal.

Now assume that r does not divide $d(\chi_g)$. Since r does divide $d(\chi_r)$, it follows from Lemma 4.2 that r divides $d(\chi_g + \chi_r)$. Now, g and $d(\chi_r)$ are relatively prime, since every prime dividing $d(\chi_r)$ is at most u (based on the work in Section 3), but every prime dividing g is greater than u . On the other hand, g divides $d(\chi_g)$. So applying Lemma 4.2 to each prime [power] factor of g , we get that g must divide $d(\chi_g + \chi_r)$.

But if both g and r divide $d(\chi_g + \chi_r)$, then h must divide $d(\chi_g + \chi_r)$. Thus, $\chi_h \leq \chi_g + \chi_r = [\chi_g + \chi_r]$.

We've now proven that $\chi_g \leq \chi_h \leq [\chi_g + \chi_r]$; that is only possible if $\chi_h = [\chi_g + \alpha]$ for some $\alpha \leq \chi_r$. Thus $\chi_h = \chi_g + \alpha$. But then $\alpha = \chi_h - \chi_g$, and since r divides $d(\chi_h)$ but not $d(\chi_g)$, r must divide $d(\alpha)$ by Lemma 4.2. Since r divides $d(\alpha)$, we have $\alpha \geq \chi_r$; thus, $\alpha = \chi_r$, and we're done. \square

Corollary 4.4. *For every natural number h , there is a unique finite set $Q(h)$ of prime [powers] where $\chi_h = \sum_{r \in Q(h)} \chi_r$. (When there is ambiguity about the choice of field On_p , we will write $Q_p(h)$ for $Q(h)$.) Every $r \in Q(h)$ divides h and is relatively prime to $[h/r]$. Finally, if $h > 1$ and u is the largest prime dividing h , then the largest [power] of u dividing h belongs to $Q(h)$.*

This corollary follows from Theorem 4.3 by induction; we can rewrite χ_h as either χ_g or $\chi_g + \chi_r$, and if g is not a prime [power], then we can rewrite χ_g in a similar fashion, and so on.

Next, a lemma generalizing Lemma 3.4 from [6].

Lemma 4.5. *If $\chi \in [\omega^{\omega^\omega}]$, then the multiplicative group of the field $\omega(\chi)$ is generated by the elements $\chi + m$, $m \in \omega$.*

The proof is identical to the proof in [6]; the proof does not depend on the value of p .

Proof. Let $F(x) = \sum f_i x^i \in \omega[x]$ (where each $f_i \in \omega$) be the irreducible polynomial of χ in $\omega[x]$. Let β be any nonzero element of $\omega(\chi)$; assume $\beta = \sum g_j \chi^j$, where each $g_j \in \omega$. Let μ be the subfield of ω generated all by the coefficients f_i and g_j . Then the polynomials $G(x) = \sum g_j x^j$ and $F(x)$ are both contained in $\mu[x]$. And since $F(x)$ is irreducible in $\mu[x]$, and since $G(x)$ (a nonzero polynomial) has smaller degree than $F(x)$, $G(x)$ and $F(x)$ must be relatively prime in $\mu[x]$.

We may then apply Kornblum-Artin's analogue of Dirichlet's theorem on primes in arithmetic progressions (which appears on pg. 94 of [1] and on pg. 39 of [7]); if $t \in \omega$ is sufficiently large, then there exists a monic polynomial $H(x) \in \mu[x]$ where $H(x) \equiv G(x) \pmod{F(x)}$

(hence, $H(\chi) = \beta$). If t is chosen to be a [power] of 2, then since ω is quadratically closed, $H(x)$ is a product of linear factors in $\omega[x]$. If $H(x) = \prod_i (x + m_i)$ (where each $m_i \in \omega$), then $\beta = H(\chi) = \prod_i (\chi + m_i)$. So all nonzero elements of $\omega(\chi)$ are products of elements of form $\chi + m$ ($m \in \omega$), and that proves the theorem. \square

We will use Lemma 4.5 to prove the next theorem, a generalization to On_p of Theorem 3.1 from [6]. It allows us to fully classify the elements α_u up to a finite term.

For any prime $u \neq p$, let ζ_u be a primitive u th root of unity in $[\omega^{\omega^\omega}]$. Let $f(u) = d(\zeta_u)$. That is, $f(u)$ is the natural number where u is a primitive divisor of $[p^{f(u)} - 1]$. (Thus, $f(u)$ is a divisor of $[u - 1]$.)

Theorem 4.6. *For any prime number $u \neq p$, there exist natural numbers m and m' where $\alpha_u = [\chi_{f(u)} + m] = \chi_{f(u)} + m'$.*

Here, m is the "excess" of α_u over $\chi_{f(u)}$.

Proof. By definition, α_u is not a u th power in the field χ_u . Let $F = p(\alpha_u)$, and let F^* be the multiplicative group of the field F . Consider the group homomorphism $\theta_1 : F^* \Rightarrow F^*$, where $\theta_1(a) = a^u$; α_u is not in the image of this map, so it is not surjective. So it is not injective either; there are elements of F whose u th power is 1, and the only such elements are the primitive u th-roots of unity. Thus, $\zeta_u \in F = p(\alpha_u)$. So $d(\alpha_u)$ is a multiple of $d(\zeta_u) = f(u)$, so $\alpha_u \geq \chi_{f(u)}$.

On the other hand, since $d(\chi_{f(u)})$ is divisible by $f(u)$, ζ_u must be an element of $p(\chi_{f(u)})$. Let $G = p(\chi_{f(u)})$, and let G^* be the multiplicative group of the field G . Consider the group homomorphism $\theta_2 : G^* \Rightarrow G^*$, where $\theta_2(a) = a^u$; since $\zeta_u \in G^*$, this map cannot be injective. So it is not surjective either; there is some $\beta \in G$ that is not a u th power in G . Now, we have $G = p(\chi_{f(u)}) \subseteq \chi_u$, and all extensions of $p(\chi_{f(u)})$ contained within χ_u are of degree less than u . So, β is not a u th power in χ_u .

Note that $\beta \in \omega(\chi_{f(u)})$; by Lemma 4.5, we can write β as a product of elements of the form $\chi_{f(u)} + m$, $m \in \omega$. Since β is not a u th power in χ_u , at least one of the factors $\chi_{f(u)} + m_0$ is also not a u th power in χ_u . Thus, $\alpha_u \leq \chi_{f(u)} + m_0$.

Finally, write $\chi_{f(u)}$ as $[\lambda + m_1]$, where λ is a limit ordinal and $m_1 \in \omega$. So, $\alpha_u \geq [\lambda + m_1]$. We have $\lambda + m = [\lambda + m]$ for all $m \in \omega$, so $\alpha_u \leq \chi_{f(u)} + m_0 = \lambda + m_1 + m_0 = [\lambda + m_2]$, where $m_2 = m_1 + m_0$. So we have $[\lambda + m_1] \leq \alpha_u \leq [\lambda + m_2]$ for some natural numbers m_1 and m_2 ; $\alpha_u = [\lambda + m_1 + m]$ for some natural number m .

From that, we can conclude the following: $\alpha_u = [[\lambda + m_1] + m] = [\chi_{f(u)} + m]$, and $\alpha_u = [\lambda + [m_1 + m]] = \lambda + [m_1 + m] = \chi_{f(p)} - m_1 +$

TABLE 1. $\alpha_u \in On_2$

u	$f(u)$	$Q(f(u))$	excess	α_u
3	2	$\{2\}$	0	2
5	4	$\{4\}$	0	$[2^2]$
7	3	$\{3\}$	1	$[2^\omega] + 1$
11	10	$\{5\}$	1	$[2^{\omega^2}] + 1$
13	12	$\{3,4\}$	0	$[2^\omega] + [2^2]$
17	8	$\{8\}$	0	$[2^4]$
19	18	$\{9\}$	4	$[2^{\omega \cdot 3}] + 4$
23	11	$\{11\}$	1	$[2^{\omega^4}] + 1$
29	28	$\{7,4\}$	0	$[2^{\omega^3}] + [2^2]$
31	5	$\{5\}$	1	$[2^{\omega^2}] + 1$
37	36	$\{9,4\}$	0	$[2^{\omega \cdot 3}] + [2^2]$
41	20	$\{5\}$	1	$[2^{\omega^2}] + 1$
43	14	$\{7\}$	1	$[2^{\omega^3}] + 1$

TABLE 2. $\alpha_u \in On_3$

u	$f(u)$	$Q(f(u))$	excess	α_u
2	1	\emptyset	2	2
5	4	$\{4\}$	1	$[3^2] + 1$
7	6	$\{3,2\}$	0	$[3^\omega] + 3$
11	5	$\{5\}$	1	$[3^{\omega^2}] + 1$
13	3	$\{3\}$	0	$[3^\omega]$
17	16	$\{16\}$	1	$[3^8] + 1$
19	18	$\{9,2\}$	0	$[3^{\omega \cdot 3}] + 3$
23	11	$\{11\}$	1	$[3^{\omega^4}] + 1$
29	28	$\{7,4\}$	0	$[3^{\omega^3}] + [3^2]$
31	30	$\{5,3\}$	0	$[3^{\omega^2}] + [3^\omega]$
37	18	$\{9,2\}$	0	$[3^{\omega \cdot 3}] + 3$
41	8	$\{8\}$	1	$[3^4] + 1$
43	42	$\{7\}$	1	$[3^{\omega^3}] + 1$

$[m_1 + m] = \chi_{f(p)} + m'$, for some natural number m' . That completes the proof. \square

Using these theorems, and with the aid of Mathematica, I was able to assemble the Tables 1 through 5, which contain the elements $\alpha_u \in On_p$ for $u \leq 43, p \leq 11$. (The table for On_2 first appeared in [6].)

TABLE 3. $\alpha_u \in On_5$

u	$f(u)$	$Q(f(u))$	excess	α_u
2	1	\emptyset	2	2
3	2	$\{2\}$	1	$5 + 1$
7	6	$\{3\}$	1	$[5^\omega] + 1$
11	5	$\{5\}$	0	$[5^{\omega^2}]$
13	4	$\{4\}$	1	$[5^2] + 1$
17	16	$\{16\}$	1	$[5^8] + 1$
19	9	$\{9\}$	1	$[5^{\omega \cdot 3}] + 1$
23	22	$\{11, 2\}$	0	$[5^{\omega^4}] + 5$
29	14	$\{7\}$	1	$[5^{\omega^3}] + 1$
31	3	$\{3\}$	1	$[5^\omega] + 1$
37	36	$\{9, 4\}$	0	$[5^{\omega \cdot 3}] + [5^2]$
41	20	$\{5, 4\}$	0	$[5^{\omega^2}] + [5^2]$
43	42	$\{7\}$	1	$[5^{\omega^3}] + 1$

TABLE 4. $\alpha_u \in On_7$

u	$f(u)$	$Q(f(u))$	excess	α_u
2	1	\emptyset	3	3
3	1	\emptyset	2	2
5	4	$\{4\}$	1	$[7^2] + 1$
11	10	$\{5\}$	1	$[7^{\omega^2}] + 1$
13	12	$\{3, 4\}$	0	$[7^\omega] + [7^2]$
17	16	$\{16\}$	1	$[7^8] + 1$
19	3	$\{3\}$	1	$[7^\omega] + 1$
23	22	$\{11\}$	1	$[7^{\omega^4}] + 1$
29	7	$\{7\}$	0	$[7^{\omega^3}]$
31	15	$\{5, 3\}$	0	$[7^{\omega^2}] + [7^\omega]$
37	9	$\{9\}$	1	$[7^{\omega \cdot 3}] + 1$
41	40	$\{5, 8\}$	0	$[7^{\omega^2}] + [7^4]$
43	6	$\{3, 2\}$	3	$[7^\omega] + 7 + 3$

5. CONCLUSION: THOUGHTS ABOUT On_0

To conclude, let's consider how we might turn the ordinals into On_0 , a Field of characteristic zero. The inductive construction from Section 1 works just as well in the characteristic zero case, so all that needs to be determined is how to define the smallest field ϕ_0 (which will be isomorphic to \mathbb{Q} , the field of rationals).

TABLE 5. $\alpha_u \in On_{11}$

u	$f(u)$	$Q(f(u))$	excess	α_u
2	1	\emptyset	2	2
3	2	$\{2\}$	1	$11 + 1$
5	1	\emptyset	2	2
7	3	$\{3\}$	1	$[11^\omega] + 1$
13	12	$\{3,4\}$	0	$[11^\omega] + [11^2]$
17	16	$\{16\}$	1	$[11^8] + 1$
19	3	$\{3\}$	1	$[11^\omega] + 1$
23	22	$\{11,2\}$	0	$[11^{\omega^4}] + 11$
29	28	$\{7,4\}$	0	$[11^{\omega^3}] + [11^2]$
31	30	$\{5,3\}$	0	$[11^{\omega^2}] + [11^\omega]$
37	6	$\{3\}$	1	$[11^\omega] + 1$
41	40	$\{5,8\}$	0	$[11^{\omega^2}] + [11^4]$
43	7	$\{7\}$	1	$[11^{\omega^3}] + 1$

We'll fill in the addition table first, then the multiplication table; for each possible sum or product, we'll choose the smallest ordinal that still allows us to construct a Field of characteristic zero. First of all, we let $0 + 0 = 0$; that forces 0 to be the additive identity, so we have $0 + \alpha = \alpha$ for all ordinals α . Next, we cannot have $1 + 1 = 0$ (or else we no longer have characteristic zero) or $1 + 1 = 1$ (since 1 is not the additive identity), but we can (and must) have $1 + 1 = 2$. We then have $1 + 2 = 3$, $1 + 3 = 4$, and so on; $1 + \alpha = [\alpha + 1]$ for all $\alpha \in \omega$. But that forces the rest of the addition table for all finite ordinals: for $\alpha, \beta \in \omega$, we have $\alpha + \beta = [\alpha + \beta]$. We just have ordinary addition (and hence, ordinary multiplication) within ω .

Now, the next sum to determine is the sum of ω and 1. We can have $\omega + 1 = 0$; ω can play the role of -1 . Thus, $\omega + 2 = 1$, $\omega + 3 = 2$, etc. The next undetermined sum is then $\omega + \omega$; this cannot equal ω or anything in ω , so we must have $\omega + \omega = [\omega + 1]$. So $[\omega + 1] = -2$, and similarly $[\omega + 2] = -3$, and so on. We thus obtain our first group: $[\omega \cdot 2]$ is isomorphic to \mathbb{Z} , the ring of integers.

Since $[\omega \cdot 2]$ is a group, for every $\alpha \in [\omega \cdot 2]$, $[\omega \cdot 2] + \alpha$ cannot be an element of $[\omega \cdot 2]$. So making the simplest choices at every step, we let $[\omega \cdot 2] + 1 = [\omega \cdot 2 + 1]$, $[\omega \cdot 2] + 2 = [\omega \cdot 2] + 2$, and so on; we then have $[\omega \cdot 2] + \alpha = [\omega \cdot 2 + \alpha]$ for all $\alpha \in [\omega \cdot 2]$. The next sum to consider is $[\omega \cdot 2] + [\omega \cdot 2]$. This sum cannot be 0, but it can (and thus must) be 1, since $[\omega \cdot 2]$ may play the role of $1/2$. We thus get $[\omega \cdot 4]$ as our next group, consisting of all halves of integers. We similarly get

$[\omega \cdot 4] = 1/4$, $[\omega \cdot 8] = 1/8$, and so on; $[\omega^2]$ is our next ring, the ring of dyadic rationals.

We then have $[\omega^2] + \alpha = [\omega^2 + \alpha]$ for all $\alpha \in [\omega^2]$, so the next sum to consider is $[\omega^2] + [\omega^2]$. This cannot be anything in $[\omega^2]$, since all elements of $[\omega^2]$ already have halves; thus, we must have $[\omega^2] + [\omega^2] = [\omega^2 \cdot 2]$. However, we can (and must) have $[\omega^2] + [\omega^2 \cdot 2] = 1$, letting $[\omega^2]$ play the role of $1/3$. Similarly, we have $[\omega^2 \cdot 3] = 1/9$, $[\omega^2 \cdot 9] = 1/27$, etc. Our next ring is then $[\omega^3] = 1/5$, and we then have $[\omega^4] = 1/7$, $[\omega^5] = 1/11$, and so on. Our first field is thus $[\omega^\omega]$, which is isomorphic to \mathbb{Q} .

With the first field ϕ_0 thus constructed, we then use the same construction as for On_p to construct all of On_0 . Unfortunately, further analysis of On_0 would seem to be very difficult. We were able to obtain a nearly complete analysis of On_p below the first transcendental, mostly because all elements below $[\omega^{\omega^\omega}]$ are contained in finite fields. But there are no finite fields in On_0 (or in any field of characteristic zero, for that matter). I would imagine that finding the first transcendental in On_0 would be as difficult as finding the second transcendental in any On_p . So we won't analyze On_0 any further.

REFERENCES

- [1] E. Artin, *Collected Papers*, Addison Wesley, Reading, 1965.
- [2] J.H. Conway, *On Numbers and Games 2nd edition*, A K Peters, Ltd., Natick, MA, 2001.
- [3] J.M. DiMuro, *On Prime Power Order Elements of General Linear Groups*, pending.
- [4] F. Laubie, *A recursive definition of p-ary addition without carry*, Journal de Théorie des Nombres de Bordeaux, **11** (1999), 307-315
- [5] H.W. Lenstra, Jr., *Nim multiplication*, I.H.E.S., Bures-sur-Yvette.
- [6] H.W. Lenstra, Jr., *On the Algebraic Closure of Two*, Proc. Kon. Ned. Akad. Wet. Series A **80**, 389-396.
- [7] M.I. Rosen, *Number Theory in Function Fields*, Springer-Verlag, New York, 2002.

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